

On the Separability of GDoF Region for Parallel Gaussian TIN Optimal Interference Networks

Hua Sun and Syed A. Jafar

Center for Pervasive Communications and Computing (CPCC)
University of California Irvine, Irvine, CA 92697

Abstract—It has been shown recently by Sun et al. that in a K user parallel Gaussian interference network, if over each sub-channel, for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user (all signal strengths measured in dB scale), then separate coding over each sub-channel and treating interference as noise (TIN) is sufficient to achieve the *sum* generalized degrees of freedom (GDoF), subject to a mild invertibility condition [1]. In this work, we show that the *weighted* sum GDoF is similarly separable, i.e., separate coding and TIN is sufficient to achieve the weighted sum GDoF, subject to a similar mild invertibility condition. This is proved by translating the weighted GDoF optimization problem to the sum GDoF problem of a class of *compound* parallel Gaussian interference networks, giving rise to new weighted GDoF outer bounds that are strictly stronger than what is implied by the sum GDoF bounds obtained previously.

I. INTRODUCTION

There is much recent interest in the capacity of Gaussian interference networks in the regime where the simplest scheme, where each transmitter uses point to point Gaussian codebook with power control and each receiver “treats interference as noise” (TIN), is information theoretically optimal. The information theoretic optimality of TIN is considered from the exact capacity perspective in [2], [3], [4], [5] where such a TIN optimal regime is identified. However, either due to the difficulty of characterizing the exact capacity or because exact optimality of TIN is indeed rare, the regime that is identified so far is very small, limiting its significance in practice. On the other hand, one would expect a much larger regime where TIN is approximately optimal. Such a regime may be useful for broader insights into the optimality of TIN and may be also more tractable. Based on this intuition, recent work has explored the optimality of TIN by relaxing the metric from exact capacity to approximate capacity, specifically in terms of the generalized degrees of freedom (GDoF).

The starting point of this work is the result by Geng et al. in [6], where an elegant TIN optimality condition is identified under which TIN is sufficient to achieve the whole GDoF region (also the capacity region to a constant gap). The TIN optimality condition is that in a K user Gaussian interference network, for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user

(all signal strengths measured in dB scale). More recently, [1] considers the K user parallel Gaussian TIN optimal interference network, whereby each sub-channel satisfies the TIN optimality condition. It is shown that the *sum* GDoF of the parallel network can be achieved by separate coding and TIN over each sub-channel, subject to a mild invertibility condition. The main motivation of this work is to go beyond the sum GDoF to the GDoF region, and in particular to answer the following question: *Is separate TIN over each sub-channel still optimal for the GDoF region of the K user parallel network collectively, given that each sub-channel individually is TIN optimal?*

As the GDoF region can be described by the weighted sum GDoF for all choices of weights, we focus on the weighted sum GDoF problem. We show that any weighted GDoF optimization (that is, any supporting hyperplane of the GDoF region) of the parallel network can, without loss of optimality, be solved separately over each sub-channel and the optimal value for each sub-channel can be achieved by TIN, subject again to a mild invertibility condition similar to the one identified in [1]. The main difficulty which limited our previous work to sum GDoF is that, unlike the case without parallel channels studied by Geng et al. where it is shown that sum-GDoF bounds are sufficient to characterize the entire GDoF region, with parallel channels it turns out that sum GDoF bounds alone are not sufficient to describe the region achieved by separate TIN (an example is given in [1]). This is essentially because the GDoF region is not polymatroidal. The new insight that allows us to overcome this difficulty in this work is that the weighted GDoF optimization problem can be translated into the sum GDoF problem of a class of compound parallel Gaussian interference networks. For any given vector of weights, we are able to define a corresponding compound parallel Gaussian interference network such that the weighted-sum GDoF of the original parallel network is equal to the sum GDoF of this compound parallel network. This allows us to use the insights from [1] to find the sum GDoF value and show that it is achieved by separate TIN on each subchannel.

II. SYSTEM MODEL, DEFINITIONS, AND PRELIMINARIES

A. Gaussian Interference Network and GDoF Framework

We retain the channel model of [1], represented as follows. Consider the K user real Gaussian interference network, with

M parallel sub-channels, described as

$$Y_k^{[m]}(t) = \sum_{i=1}^K \text{sign}(\tilde{h}_{ki}^{[m]}) \sqrt{P\alpha_{ki}^{[m]}} X_i^{[m]}(t) + Z_k^{[m]}(t) \quad (1)$$

$$\forall k \in [K] \triangleq \{1, 2, \dots, K\}, m \in [M] \triangleq \{1, 2, \dots, M\}$$

where at each time index t , over the m -th sub-channel, $X_i^{[m]}(t)$ is the transmitted symbol from Transmitter i , $Y_k^{[m]}(t)$ is the received signal observed at Receiver k , $\text{sign}(\tilde{h}_{ki}^{[m]}) \in \{-1, 1\}$ indicates the real channel from Transmitter i to Receiver k is negative or positive, $\sqrt{P\alpha_{ki}^{[m]}}$ is its channel magnitude, and $Z_k^{[m]}(t) \sim \mathcal{N}(0, 1)$ is the additive white Gaussian noise (AWGN) at Receiver k . Noise processes are i.i.d over time and sub-channels. All symbols are real. We call the exponent $\alpha_{ki}^{[m]}$ the channel strength level and as done in [6], [1], we assume $\alpha_{ki}^{[m]}$ values are non-negative (the negative values are mapped to zero). All channel coefficients are fixed over time. Perfect channel knowledge is available everywhere.

At Transmitter i , an independent message W_i uniformly distributed over the index set $\{1, 2, \dots, 2^{nR_i}\}$ is mapped to the codeword $[X_i^{[1]}(1), \dots, X_i^{[M]}(1), X_i^{[1]}(2), \dots, X_i^{[M]}(n)]$ over n time slots, and is subject to the average power constraint, $\frac{1}{n} \sum_{t=1}^n \sum_{m=1}^M |X_i^{[m]}(t)|^2 \leq 1$. At Receiver k , the received signal $[Y_k^{[1]}(1), \dots, Y_k^{[M]}(1), Y_k^{[1]}(2), \dots, Y_k^{[M]}(n)]$ is used to produce the estimate \hat{W}_k of the message W_k . The probability of error for Receiver k is given by the probability that \hat{W}_k is not equal to W_k . A rate tuple $\mathbf{R}(P) = (R_1(P), R_2(P), \dots, R_K(P))$ is said to be achievable if we have an encoding and decoding mapping such that the probability of error for each receiver approaches zero as n approaches infinity. The capacity region $\mathcal{C}(P)$ is the closure of the set of all achievable rate tuples.

We define the (convex) GDoF region as the collection of weighted sum GDoF problems in (2), shown at the top of the next page, and the sum GDoF value as $\mathcal{D}_\Sigma = \max_{\mathcal{D}} \sum_{k=1}^K d_k$.

If the following TIN optimality condition is satisfied

$$\alpha_{ii}^{[m]} \geq \max_{j:j \neq i} \{\alpha_{ji}^{[m]}\} + \max_{k:k \neq i} \{\alpha_{ik}^{[m]}\}, \quad \forall i, j, k \in [K], m \in [M]$$

the network (sub-channel) is referred to as a TIN optimal network (sub-channel) and in this work, we focus exclusively on TIN optimal networks, i.e., the condition holds throughout.

B. Weighted Directed Graph Representation

A directed graph is defined for each sub-channel of the parallel interference network. As all definitions and notations to be introduced are associated with one specific sub-channel, the sub-channel index superscript is omitted in this subsection for compactness. The directed graph representation of one sub-channel of the parallel interference network consists of K vertices, V_1, V_2, \dots, V_K , one for each user. Since the vertices correspond directly to users, we will also refer to them as users. For all $(i, j) \in [K] \times [K]$, there is a directed edge e_{ij} from user j to user i , with weight $w(e_{ij})$ defined as $w(e_{ij}) = \alpha_{ij}$ if $i \neq j$ and $w(e_{ij}) = 0$ if $i = j$.

We are particularly interested in the notion of cycles on this directed graph. We define a cycle, π , as a cyclically ordered subset of users, without repetitions. The set of all cycles is denoted as $[\Pi]$. The cardinality of a cycle, denoted as $|\pi|$ is the number of users that it involves. $|\pi| = \sum_{V_k \in \pi} 1, \forall \pi \in [\Pi]$. A cycle with only one user is a trivial cycle. Two cycles π_p, π_q , are said to be disjoint if they contain no common user, denoted as $\pi_p \cap \pi_q = \phi$.

Introducing a slight abuse of notation in the interest of conciseness, the same cycle, π , can also be equivalently represented as a set of edges representing a closed path where no user is visited more than once. The weight of a cycle, denoted as $w(\pi)$, is the sum of the weights of all the edges traversed in completing the cycle, i.e., $w(\pi) = \sum_{e_{ij} \in \pi} w(e_{ij}), \forall \pi \in [\Pi]$. Note that the weight of a trivial cycle is zero. Intuitively, the weight of a cycle is the accumulation of the strengths of interference terms encountered in the cycle.

Cyclic Partition: A subset of the set of all cycles, $\Pi \subset [\Pi]$, is said to be a cyclic partition if $\pi_p \cap \pi_q = \phi, \forall \pi_p, \pi_q \in \Pi$ and $\sum_{\pi \in \Pi} |\pi| = K$. In other words, a cyclic partition is a disjoint cyclic cover of the K users.

Cyclic Partition Bound: For any cyclic partition Π , define the corresponding cyclic partition bound, \mathcal{D}_Σ^Π , as $\sum_{k=1}^K d_k \leq \sum_{k=1}^K \alpha_{kk} - w(\Pi)$, where $w(\Pi) = \sum_{\pi \in \Pi} w(\pi)$ is the net weight of the cyclic partition, representing the total interference encountered in this partition. We also call $w(\pi)$ cycle bound, then cyclic partition bound is merely a collection of disjoint cycle bounds. Since there are many cyclic partitions, each of which gives rise to a cyclic partition bound, let us denote the tightest of these bounds as the *best cyclic partition bound*, $\mathcal{D}_\Sigma^{\Pi^*}$. A cyclic partition that produces the best cyclic partition bound is labeled an *optimal* cyclic partition, and denoted by Π^* . For example, when $K = 6$, one possible cyclic partition is $\Pi = \{\{1, 3, 5\}, \{4, 2\}, \{6\}\}$ which decomposes the users into three cycles, such that each user is represented in exactly one cycle. The corresponding cyclic partition bound is $\sum_{k=1}^6 d_k \leq \sum_{k=1}^6 \alpha_{kk} - (\alpha_{13} + \alpha_{35} + \alpha_{51}) - (\alpha_{42} + \alpha_{24}) - (0)$.

Participating Edge: Edge e_{ij} is a participating edge for the cyclic partition Π if $i \neq j$ and $e_{ij} \in \pi$ for some $\pi \in \Pi$.

Cyclic Predecessor: Under cyclic partition Π , the cyclic predecessor for user k is user $\Pi(k)$, if $e_{\Pi(k)k}$ is a participating edge for Π . Note that if user k belongs to a trivial cycle in Π then $\Pi(k) = \phi$.

Participating Input and Output Levels ($X_{i,u}, Y_{k,u}$): For any sub-channel, we define participating input levels $X_{i,u} \triangleq 0.X_{i,(1)}, \dots, X_{i,(n_{\Pi(i)}i)}, n_{\Pi(i)}i = \lfloor \frac{1}{2} \alpha_{\Pi(i)i} \log_2 P \rfloor$ to be the bits levels that are below the decimal point, sent from Transmitter i and observed at its predecessor Receiver $\Pi(i)$. The received signal levels resulting from all interfering $X_{i,u}$ plus additive Gaussian noise are defined as the participating output levels $Y_{k,u} \triangleq \sum_{i=1, i \neq k}^K \text{sign}(\tilde{h}_{ki}) \sqrt{P\alpha_{ki}} X_{i,u} + Z_k$.

Invertibility: One sub-channel is said to be invertible within bounded noise distortion if the mapping from $\mathbf{X}_u \triangleq (X_{1,u}, \dots, X_{K,u})$ to $\mathbf{Y}_u \triangleq (Y_{1,u}, \dots, Y_{K,u})$ is invertible for an optimal cyclic partition Π^* . Mathematically, we require $H(\mathbf{X}_u | \mathbf{Y}_u) = o(\log(P))$.

$$\mathcal{D} = \left\{ (d_1, \dots, d_K) \in \mathbb{R}_+^K : \forall (\mu_1, \dots, \mu_K) \in \mathbb{R}_+^K, \mu_1 d_1 + \dots + \mu_K d_K \leq \limsup_{P \rightarrow \infty} \left[\sup_{\mathbf{R}(P) \in \mathcal{C}(P)} [\mu_1 R_1(P) + \dots + \mu_K R_K(P)] \frac{1}{\log P} \right] \right\} \quad (2)$$

$$\begin{aligned} \bar{Y}_{i;p;l}^{[m]} &= \text{sign}(\tilde{h}_{ii}^{[m]}) \sqrt{P \alpha_{ii}^{[m]}} \bar{X}_{i;p}^{[m]} + \sum_{j=1, j \neq i}^K \text{sign}(\tilde{h}_{ij}^{[m]}) \sqrt{P \alpha_{ij}^{[m]}} \bar{X}_{j;a_j}^{[m]} + \bar{Z}_{i;p;l}^{[m]}, \forall i \in [K], m \in [M], \quad a_{i+1} = \lceil \frac{l}{\mu_{i+2}\mu_{i+3} \dots \mu_{i-1}} \rceil, \\ a_j &= \lceil \frac{l - \sum_{q=i+1}^{j-1} (a_q - 1) \mu_{q+1} \mu_{q+2} \dots \mu_{i-1}}{\mu_{j+1} \mu_{j+2} \dots \mu_{i-1}} \rceil, j = \{i+2, i+3, \dots, i-2\}, \quad a_{i-1} = l - \sum_{q=i+1}^{i+K-2} (a_q - 1) \mu_{q+1} \mu_{q+2} \dots \mu_{i-1} \end{aligned} \quad (3)$$

Remark: Participating inputs and output levels ($X_{i,u}, Y_{k,u}$) and associated invertibility properties ($H(\mathbf{X}_u | \mathbf{Y}_u)$) are defined respective to some cyclic partition, though this dependency is not explicitly shown in the notation. The referred cyclic partition is understood from the context when these notations appear. We mention that the invertibility property is mild. For a detailed discussion, we refer to Section 5.3.2 in [1].

To consider the weighted sum GDoF problem, we translate it to the sum GDoF problem of another network, defined next. Suppose we are considering maximizing $\mu_1 d_1 + \dots + \mu_K d_K$ for positive integer weights $\mu_i \in \mathbb{Z}_+, i \in [K]$.

C. Replicated Compound Network

In the replicated compound network, Transmitter i is replicated $\mu_i, i \in [K]$ times. Each replicated transmitter $k \in [K]$ has $\prod_{i=1, i \neq k}^K \mu_i$ compound receivers. An example with $K = 3, (\mu_1, \mu_2, \mu_3) = (3, 2, 1)$ is shown in Figure 1.

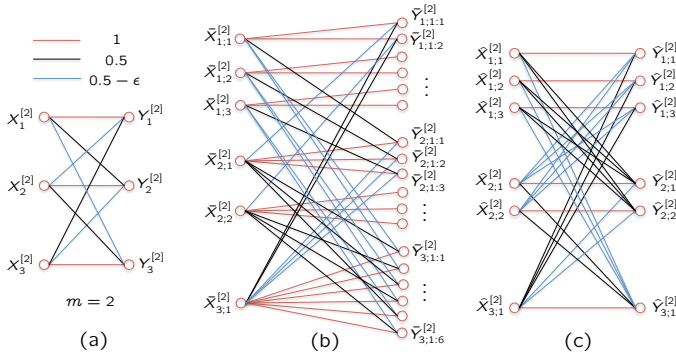


Fig. 1. (a) The second sub-channel of an network, (b) the corresponding replicated compound sub-channel and (c) the corresponding auxiliary sub-channel (to be used in later proof). The channel strength level is indicated by the color of each link. To avoid cluttering the figure, in (b), only the channels to the receivers associated with the first replicate of each transmitter are shown.

The input output relationship of the replicated compound network is described as (3), shown at the top of this page, where over the m -th sub-channel, the transmitted symbol from the p -th replicate of Transmitter j is denoted as $\bar{X}_{j;p}^{[m]}, \forall p \in \{1, \dots, \mu_j\}$, the received signal and AWGN observed at the l -th compound receiver associated with the p -th replicate of Transmitter i are represented by $\bar{Y}_{i;p;l}^{[m]}$ and $\bar{Z}_{i;p;l}^{[m]}, \forall p \in$

$\{1, \dots, \mu_i\}, l \in \{1, \dots, \prod_{j=1, j \neq i}^K \mu_j\}$, respectively. The indices are interpreted modulo K . We set $\bar{Z}_{i;p;l}^{[m]} = Z_i^{[m]}$. In words, the network is designed such that each compound receiver sees one and only one interference from each interfering replicated transmitter, without repetition.

For the replicated compound network, each transmitter has an independent message for its compound receivers. Capacity region and associated notions are defined similar to those of the original network. Specifically, we denote the GDoF tuple as $(\bar{d}_{1;1}, \dots, \bar{d}_{1;\mu_1}, \bar{d}_{2;1}, \dots, \bar{d}_{K;\mu_K})$ and the sum GDoF value as $\bar{\mathcal{D}}_\Sigma = \max_{\bar{\mathcal{D}}} (\sum_{k=1}^K \sum_{j=1}^{\mu_k} \bar{d}_{k;j})$, where $\bar{\mathcal{D}}$ is the GDoF region of the replicated compound network.

The usage of the replicated compound network lies in the following lemma, which reduces the weighted sum GDoF problem to the sum GDoF problem.

Lemma 1: The weighted GDoF of the original network is equal to the sum GDoF of the replicated compound network, i.e., $\mu_1 d_1 + \dots + \mu_K d_K = \bar{\mathcal{D}}_\Sigma$.

The proof is based on using the coding schemes of the original network in the replicated compound network and vice versa. Details are presented in Appendix A.

After this reduction, the problem is still not immediately solved, as the sum GDoF of a parallel TIN optimal compound interference network is generally open. But our constructed replicated compound network contains some symmetry that allows us to solve its sum GDoF value analytically. To find $\bar{\mathcal{D}}_\Sigma$, it is helpful to introduce the weighted directed graph representation for the replicated compound network, similar to that of the original network.

We have one directed graph for each compound state of the replicated sub-channel. One compound state is defined by keeping only one compound receiver for each replicated transmitter. Consider one sub-channel and one compound state where the received signals are $\bar{Y}_{1;1;l_{1;1}}, \dots, \bar{Y}_{1;\mu_1;l_{1;\mu_1}}, \bar{Y}_{2;1;l_{2;1}}, \dots, \bar{Y}_{K;\mu_K;l_{K;\mu_K}}$. Note that the sub-channel index superscript is omitted for brevity. The directed graph consists of $\sum_{k=1}^K \mu_k$ vertices, $\bar{V}_{1;1}, \dots, \bar{V}_{1;\mu_1}, \bar{V}_{2;1}, \dots, \bar{V}_{K;\mu_K}$, one for each user. For all $i, j \in [K], p_i \in \{1, \dots, \mu_i\}, p_j \in \{1, \dots, \mu_j\}$, there is a directed edge $\bar{e}_{i;p_i,j;p_j}$ from user $(j; p_j)$ to user $(i; p_i)$, with weight $w(\bar{e}_{i;p_i,j;p_j})$ defined as $w(\bar{e}_{i;p_i,j;p_j}) = \alpha_{ij}$ if $i \neq j, a_j = p_i$ and 0 otherwise, where a_j is defined according to the index of the compound receiver $l_{i;p_i}$. In words, the weight of an edge

is equal to the strength of the interfering link.

A cycle, $\bar{\pi}$, is a cyclically ordered subset of users, without repetitions. The set of all cycles is denoted as $[\bar{\Pi}]$. The cardinality of a cycle, denoted as $|\bar{\pi}|$ is the number of users that it involves. $\bar{\pi}$, can also be equivalently represented as a set of edges representing a closed path where no user is visited more than once. The weight of a cycle, denoted as $w(\bar{\pi})$, is the sum of the weights of all the edges traversed in completing the cycle.

Cyclic Partition: A subset of the set of all cycles, $\bar{\Pi} \subset [\bar{\Pi}]$, is said to be a cyclic partition if $\bar{\pi}_p \cap \bar{\pi}_q = \emptyset$, $\forall \bar{\pi}_p, \bar{\pi}_q \in \bar{\Pi}$, $\sum_{\bar{\pi} \in \bar{\Pi}} |\bar{\pi}| = \sum_{k=1}^K \mu_k$.

Cyclic Partition Bound: For any cyclic partition $\bar{\Pi}$, define the corresponding cyclic partition bound, $\bar{\mathcal{D}}_{\Sigma}^{\bar{\Pi}}$, as $\sum_{k=1}^K \sum_{j=1}^{\mu_k} \bar{d}_{k,j} \leq \sum_{k=1}^K \mu_k \alpha_{kk} - w(\bar{\Pi})$, where $w(\bar{\Pi}) = \sum_{\bar{\pi} \in \bar{\Pi}} w(\bar{\pi})$.

We denote the tightest of cyclic partition bounds for all cyclic partitions as the *best cyclic partition bound*, $\bar{\mathcal{D}}_{\Sigma}^{\bar{\Pi}^*}$. A cyclic partition that produces the best cyclic partition bound is labeled an *optimal cyclic partition*, and denoted by $\bar{\Pi}^*$.

Participating Edge: Edge $\bar{e}_{i_{p_i} j_{p_j}}$ is a participating edge for the cyclic partition $\bar{\Pi}$ if $i \neq j$ and $\bar{e}_{i_{p_i} j_{p_j}} \in \bar{\pi}$ for some $\bar{\pi} \in \bar{\Pi}$.

Cyclic Predecessor: Under cyclic partition $\bar{\Pi}$, the cyclic predecessor for user $(k; p_k)$ is user $\bar{\Pi}(k; p_k)$, if $\bar{e}_{\bar{\Pi}(k; p_k) k_{p_k}}$ is a participating edge for $\bar{\Pi}$.

Participating Input and Output Levels ($\bar{X}_{i;p_i,u}, \bar{Y}_{k;p_k,u}$): We define the participating input levels as $\bar{X}_{i;p_i,u} \triangleq 0, \bar{X}_{i;p_i,(1)}, \dots, \bar{X}_{i;p_i,(\bar{n}_{\bar{\Pi}(i;p_i)i_{p_i}})}$, $\bar{n}_{\bar{\Pi}(i;p_i)i_{p_i}} = \lfloor \frac{1}{2} w(\bar{\Pi}(i;p_i)i_{p_i}) \log_2 P \rfloor$. We define the participating output levels as the linear combination of the interfering participating input levels plus noise, $\bar{Y}_{k;p_k,u} \triangleq \sum_{i=1, i \neq k}^K \text{sign}(\tilde{h}_{ki}) \sqrt{P^{\alpha_{ki}}} \bar{X}_{i;a_i,u} + Z_k$, where a_i is defined with the index of the compound receiver $l_{k;p_k}$.

Invertibility: The compound state of one sub-channel is said to be invertible within bounded noise distortion if the mapping from $\bar{\mathbf{X}}_u \triangleq (\bar{X}_{1;1,u}, \dots, \bar{X}_{K;\mu_K,u})$ to $\bar{\mathbf{Y}}_u \triangleq (\bar{Y}_{1;1,u}, \dots, \bar{Y}_{K;\mu_K,u})$ is invertible for an optimal cyclic partition $\bar{\Pi}^*$. Mathematically, we require $H(\bar{\mathbf{X}}_u | \bar{\mathbf{Y}}_u) = o(\log(P))$. One sub-channel is said to be invertible if the compound state whose best cyclic partition bound is the tightest among all compound states is invertible.

III. GDOF REGION

We first review the result in [1] on the sum GDoF of K user parallel Gaussian TIN optimal interference networks.

Theorem 1: (Theorem 5 in [1]) In a K user parallel Gaussian TIN optimal interference network with M sub-channels, if each sub-channel is individually invertible, then the sum GDoF value of the parallel Gaussian interference network is achieved by separate TIN over each sub-channel. Further, the sum GDoF value is given by the sum of best cyclic partition bound over each sub-channel, i.e., $\mathcal{D}_{\Sigma} = \sum_{m=1}^M \mathcal{D}_{\Sigma}^{\bar{\Pi}^{[m]*}} = \sum_{m=1}^M \left[\sum_{i=1}^K \alpha_{ii}^{[m]} - w(\bar{\Pi}^{[m]*}) \right]$.

We now move from sum GDoF to GDoF region. Interestingly, when we only have one sub-channel, the sum GDoF

bounds proved in Theorem 1 are sufficient to describe the GDoF region, which can be achieved by TIN. This is proved by Geng et al. in [6]. We restate this result in the following, as it gives us the achievable TIN region for each sub-channel by itself. For a parallel network, the Minkowski sum of the TIN region for each sub-channel is the achievable region by separate TIN.

Theorem 2: (Theorem 1 in [6]) In a K user Gaussian TIN optimal interference network with one sub-channel, power control and treating interference as noise achieve the entire GDoF region. Moreover, the GDoF region is given by all sum GDoF bounds given in Theorem 1 for all subsets of users.

The situation is different as we proceed from single sub-channel case to multiple parallel sub-channels. We have the following observation from Theorem 10 in [1], “the region described by all tight sum-GDoF bounds for all subsets of users is in general not the same as the region achievable by separate TIN over each sub-channel.” Then it is noted [1] that “either the separate TIN achievable region is not tight or we need more than sum-rate bounds.” Here we show that the latter is true, i.e., the sum GDoF bounds do not suffice.

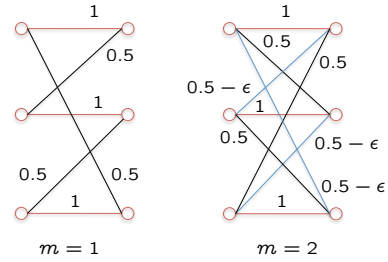


Fig. 2. A $K = 3$ user Gaussian interference network with 2 sub-channels. The channel strength level is indicated for each link. Each sub-channel is TIN optimal and invertible [1].

To be specific, we consider an example, which was introduced in [1] to show the above notes. Consider a $K = 3$ user Gaussian TIN optimal interference network with $M = 2$ sub-channels, depicted in Figure 2. We set $\epsilon = 0.01$. It is shown in [1] that each sub-channel is invertible such that Theorem 1 applies. The sum GDoF bounds proved in Theorem 1 for all subsets of users are the following.

$$d_1 \leq 1 + 1 = 2 \quad (4)$$

$$d_2 \leq 1 + 1 = 2 \quad (5)$$

$$d_3 \leq 1 + 1 = 2 \quad (6)$$

$$d_1 + d_2 \leq 1.5 + (1 + \epsilon) = 2.5 + \epsilon \quad (7)$$

$$d_2 + d_3 \leq 1.5 + (1 + \epsilon) = 2.5 + \epsilon \quad (8)$$

$$d_3 + d_1 \leq 1.5 + (1 + \epsilon) = 2.5 + \epsilon \quad (9)$$

$$d_1 + d_2 + d_3 \leq 1.5 + 1.5 = 3 \quad (10)$$

We now consider the tightest possible bound for $3d_1 + 2d_2 + d_3$, from only sum GDoF bounds, (4) - (10). That is

$$(4) + (7) + (10) \Rightarrow 3d_1 + 2d_2 + d_3 \leq 7.5 + \epsilon \quad (11)$$

However, it turns out that the tightest bound for $3d_1 + 2d_2 + d_3$ is

$$3d_1 + 2d_2 + d_3 \leq 7 + 3\epsilon \quad (12)$$

which is tighter than (11). A proof of (12) will be presented in Appendix B. Here we show how to get the RHS value of this bound. As we will show that separate TIN is sufficient to achieve any weighted sum GDoF, the tightest bound for $3d_1 + 2d_2 + d_3$ could be evaluated separately for each sub-channel and then add them up. That is

$$3d_1 + 2d_2 + d_3 = 3d_1^{[1]} + 2d_2^{[1]} + d_3^{[1]} + 3d_1^{[2]} + 2d_2^{[2]} + d_3^{[2]}$$

where $(d_1^{[m]}, d_2^{[m]}, d_3^{[m]}) \in \mathbb{R}_+^3$, $m \in \{1, 2\}$ denotes the GDoF tuple of the m -th sub-channel by itself. They satisfy the following constraints given by Theorem 2.

$$d_1^{[1]} \leq 1 \quad (13)$$

$$d_2^{[1]} \leq 1 \quad (14)$$

$$d_3^{[1]} \leq 1 \quad (15)$$

$$d_1^{[1]} + d_2^{[1]} \leq 1.5 \quad (16)$$

$$d_2^{[1]} + d_3^{[1]} \leq 1.5 \quad (17)$$

$$d_3^{[1]} + d_1^{[1]} \leq 1.5 \quad (18)$$

$$d_1^{[1]} + d_2^{[1]} + d_3^{[1]} \leq 1.5 \quad (19)$$

$$d_1^{[2]} \leq 1 \quad (20)$$

$$d_2^{[2]} \leq 1 \quad (21)$$

$$d_3^{[2]} \leq 1 \quad (22)$$

$$d_1^{[2]} + d_2^{[2]} \leq 1 + \epsilon \quad (23)$$

$$d_2^{[2]} + d_3^{[2]} \leq 1 + \epsilon \quad (24)$$

$$d_3^{[2]} + d_1^{[2]} \leq 1 + \epsilon \quad (25)$$

$$d_1^{[2]} + d_2^{[2]} + d_3^{[2]} \leq 1.5 \quad (26)$$

Thus the tightest bound on $3d_1 + 2d_2 + d_3$ based only on sum GDoF bounds for each sub-channel by itself is obtained by the sum of the following two bounds (referring to (12))

$$(13) + (16) + (19) \Rightarrow 3d_1^{[1]} + 2d_2^{[1]} + d_3^{[1]} \leq 4 \quad (27)$$

$$2 \times (23) + (25) \Rightarrow 3d_1^{[2]} + 2d_2^{[2]} + d_3^{[2]} \leq 3(1 + \epsilon) \quad (28)$$

Let us now see why linear combinations of sum GDoF bounds are loose. Note that the RHS of the sum GDoF bounds of the parallel network, (4) - (10) are the sum of the corresponding RHS of the sum GDoF of each sub-channel by itself, (13) - (19) and (20) - (26). But in order to get the tightest bound on $3d_1 + 2d_2 + d_3$, as shown in (27) and (28), we need two different linear combinations of the sum GDoF bounds of each sub-channel by itself, which is not available from the sum GDoF bounds of the parallel network. The fundamental reason is that the TIN region for each sub-channel is not polymatroidal such that their Minkowski sum is not described fully by the direct sum of corresponding inequalities for each sub-channel.

With some additional effort, we can show that the bound (12) is achievable as well, by separate TIN. In fact, this example on maximizing weighted GDoF is representative of our general result. We will show that any weighted sum GDoF problem can be solved separately over each sub-channel and the optimal value for each sub-channel is achieved by TIN. We state this result in the following theorem.

Theorem 3: In a K user parallel Gaussian TIN optimal interference network with M sub-channels, if each sub-channel of the replicated compound channel is invertible, then the weighted sum GDoF of the parallel Gaussian interference network is achieved by separate TIN over each sub-channel.

The sketch of the proof is given as follows. Lemma 1 translates the weighted sum GDoF problem of the original network to the sum GDoF problem of the replicated compound network. To find the sum GDoF of the replicated compound network, we resort to an auxiliary non-compound network (see Figure 1(c)), which contains all interfering links in all possible states. Such an auxiliary network has the property that its sum GDoF value is the same as the replicated compound network and it is further achievable by separate TIN over each sub-channel. Details are given in Appendix C.

IV. CONCLUSION

For K user parallel Gaussian TIN optimal interference networks, we show that separate TIN over each sub-channel is optimal under a mild invertibility condition, from the perspective of weighted sum GDoF, extending previous work on sum GDoF. As in the sum GDoF case, we conjecture that the invertibility condition is mild, i.e., it holds in almost all cases. Varying the weights of the weighted sum GDoF problem, we are able to cover all supporting hyperplanes of the compact convex GDoF region. But we do not claim the separability of the whole GDoF region as we are not sure the GDoF region is described by a finite number of weighted sum GDoF inequalities and we need to verify the invertibility for the replicated compound network for every possible weights (possibly infinite).

REFERENCES

- [1] H. Sun and S. A. Jafar, "On the Optimality of Treating Interference as Noise for K user Parallel Gaussian Interference Networks," *ArXiv* <http://arxiv.org/abs/1401.2692>, Jan, 2014.
- [2] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for gaussian interference channels," *IEEE Transactions on Information Theory*, vol. 55, no. 2, pp. 689–699, Feb. 2009.
- [3] A. Mottahari and A. Khandani, "Capacity bounds for the gaussian interference channel," *IEEE Transactions on Information Theory*, vol. 55, no. 2, pp. 620–643, Feb. 2009.
- [4] V. Annapureddy and V. Veeravalli, "Gaussian interference networks: Sum capacity in the low interference regime and new outer bounds on the capacity region," *IEEE Trans. on Information Theory*, pp. 3032–3050, July 2009.
- [5] X. Shang, B. Chen, G. Kramer, and H. V. Poor, "Noisy-Interference Sum-Rate Capacity of Parallel Gaussian Interference Channels," *IEEE Trans. on Inf. Theory*, vol. 57, no. 1, pp. 210–226, 2011.
- [6] C. Geng, N. Naderializadeh, S. Avestimehr, and S. Jafar, "On the Optimality of Treating Interference as Noise," *ArXiv:1305.4610*.
- [7] C. Geng and S. A. Jafar, "On the Optimality of Treating Interference as Noise: Compound Interference Networks," *e-print ArXiv:1412.2711*, Dec. 2014.

V. APPENDIX

A. Proof of Lemma 1

We show that $\mu_1 d_1 + \dots + \mu_K d_K \leq \bar{\mathcal{D}}_\Sigma$ and $\mu_1 d_1 + \dots + \mu_K d_K \geq \bar{\mathcal{D}}_\Sigma$.

$\mu_1 d_1 + \dots + \mu_K d_K \leq \bar{\mathcal{D}}_\Sigma$: To prove this claim, it suffices to prove that if the rate tuple (R_1, \dots, R_K) is achievable over the original network, then the rate tuple

$$\underbrace{(R_1, \dots, R_1)}_{\mu_1 \text{ times}}, \underbrace{(R_2, \dots, R_2)}_{\mu_2 \text{ times}}, \dots, (R_K)$$

is achievable over the replicated compound network. This is proved by using the encoding/decoding mappings of the original network in the replicated compound network. Suppose we are given a sequence of encoding and decoding mappings such that (R_1, \dots, R_K) is achievable over the original network. Then each replicated transmitter i encodes its message with the same encoding function as used by transmitter i in the original network, i.e., $\bar{X}_{i;p}^{[m]} = X_i^{[m]}, \forall i, p$. As a result, the output of the replicated compound network

$$\bar{Y}_{i;p;l}^{[m]} = \text{sign}(\tilde{h}_{ii}^{[m]}) \sqrt{P\alpha_{ii}^{[m]}} \bar{X}_{i;p}^{[m]} + \sum_{j=1, j \neq i}^K \text{sign}(\tilde{h}_{ij}^{[m]}) \sqrt{P\alpha_{ij}^{[m]}} \bar{X}_{j;a_j}^{[m]} + \bar{Z}_{i;p;l}^{[m]} \quad (29)$$

$$= \sum_{j=1}^K \text{sign}(\tilde{h}_{ij}^{[m]}) \sqrt{P\alpha_{ij}^{[m]}} X_j^{[m]} + Z_i^{[m]} \quad (30)$$

$$= Y_i^{[m]} \quad (31)$$

is statistically equivalent to the output of the original network, such that the same decoding mapping can be used to achieve the same rate. Hence we have the desired claim.

$\mu_1 d_1 + \dots + \mu_K d_K \geq \bar{\mathcal{D}}_\Sigma$: Consider the rate tuple $(\bar{R}_{1;1}, \dots, \bar{R}_{1;\mu_1}, \bar{R}_{2;1}, \dots, \bar{R}_{K;\mu_K})$ that achieves the sum capacity of the replicated compound network. Due to symmetry, we can assume that the rates of messages $\bar{W}_{i;p}$ are the same for all $p \in \{1, 2, \dots, \mu_i\}$, i.e., $\bar{R}_{i;1} = \dots = \bar{R}_{i;\mu_i} \triangleq \bar{R}_i$. To finish the proof, we only need to show that the rate tuple $(\bar{R}_1, \bar{R}_2, \dots, \bar{R}_K)$ is achievable over the original network. This follows immediately by noting that the original network is contained in the replicated compound network. Therefore, the proof is complete.

Note that from the proof, we see that not only the weighted sum GDoF of the original network is equal to the sum GDoF of the constructed network, but also the weighted sum capacity of the original network is equal to the sum capacity of the constructed network.

B. Proof of (12)

Before going to the outer bound proof of Theorem 3, we first provide a proof specifically of (12) for the network shown in Figure 2, i.e., $3d_1 + 2d_2 + d_3 \leq 7 + 3\epsilon$, in order to illustrate the main insights in a simpler setting.

Lemma 1 states that the objective is bounded by (in fact, equal to) $\bar{\mathcal{D}}_\Sigma$, the sum GDoF of the replicated compound network defined in Section II-C.

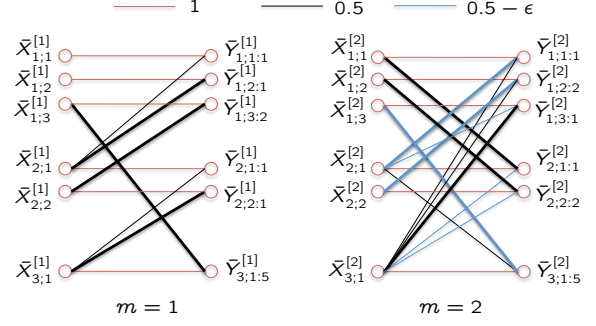


Fig. 3. The state of each sub-channel that produces the tightest bound. The channel strength level is indicated by the color of each link. The links that belong to the best cyclic partition bound are made bold.

The problem now becomes finding $\bar{\mathcal{D}}_\Sigma$. The sum GDoF of the replicated compound network can not be larger than the sum GDoF of one possible state. Then we find the tightest compound state in each sub-channel, depicted in Figure 3. The cyclic partition bound for sub-channel $m=1$ with cyclic partition $\{\{1;1\}, \{2;1,1;2\}, \{3;1,2;2,1;3\}\}$ is 4 and the cyclic partition bound for sub-channel $m=2$ with cyclic partition $\{\{2;1,1;1\}, \{2;2,1;2\}, \{3;1,1;3\}\}$ is $3(1+\epsilon)$. Note that the values match (27) and (28). The sum of they two is $7 + 3\epsilon$, as desired.

Now consider the parallel interference network with two sub-channels and each sub-channel only has one state, as shown in Figure 3. It is easily seen that the network is TIN optimal. If each sub-channel is further invertible (this is assumed to be true in Theorem 3), we can use Theorem 1 to bound its sum GDoF, which in turn bounds the weighted GDoF of the original network, i.e., $3d_1 + 2d_2 + d_3 = \bar{\mathcal{D}}_\Sigma \leq 7 + 3\epsilon$.

C. Proof of Theorem 3

In order to prove that separate TIN is sufficient to achieve the weighted sum GDoF, we prove that any weighted sum GDoF problem can be solved separately for each sub-channel and the resulting optimal value can be achieved with separate TIN. By continuity argument, we may consider only rational weights (μ_1, \dots, μ_K) of $\mu_1 d_1 + \dots + \mu_K d_K$. Further by scaling, we may restrict ourselves to integer weights. Thus without loss of generality, we assume $\mu_i, \forall i \in [K]$ are integers.

Outer Bound: We prove the outer bound first, following the proof presented in Section V-B. From Lemma 1, we have $\mu_1 d_1 + \dots + \mu_K d_K \leq \bar{\mathcal{D}}_\Sigma$, where $\bar{\mathcal{D}}_\Sigma$ is the sum GDoF of the replicated compound network defined in Section II-C. Next we proceed to find $\bar{\mathcal{D}}_\Sigma$.

The sum GDoF of the replicated compound network can not be larger than the sum GDoF of any possible state. Then naturally we want to identify the tightest compound state. Here we are faced with a challenge that was not presented in Section V-B. For such a large network, it is generally hard to identify the tightest state in each sub-channel, as there are too many choices. Instead of searching over all possible states, we consider an auxiliary network which contains all possible

cyclic partition bounds of the replicated compound network, such that we only need to consider one state essentially. Similar idea of reducing many compound states to one also appears recently in [7].

The auxiliary network is defined as follows. It contains the same transmitter set as the replicated compound network. Instead of having multiple compound receivers for each transmitter, in the auxiliary network each transmitter only has one desired receiver. We use symbols with a superscript hat for notations in the auxiliary network. Over the m -th sub-channel, the output of the receiver associated with the p -th replicated of Transmitter k is described as

$$\begin{aligned} \hat{Y}_{k;p}^{[m]} &= \text{sign}(\tilde{h}_{kk}^{[m]}) \sqrt{P_{kk}^{[m]}} \hat{X}_{k;p}^{[m]} \\ &+ \sum_{i=1, i \neq k}^K \sum_{j=1}^{\mu_i} \text{sign}(\tilde{h}_{ki}^{[m]}) \sqrt{P_{ki}^{[m]}} \hat{X}_{i;j}^{[m]} + Z_k^{[m]} \end{aligned} \quad (32)$$

In other words, the only difference is that for the replicated compound network, each receiver sees one and only one interference from each interfering replicated transmitter, while for the auxiliary network, each receiver sees *all* interference from each interfering replicated transmitter i . An illustration is given in Figure 1(c). The directed graph representation (specifically, cyclic partition bound) of the auxiliary network is defined similar to that of the replicated compound network.

The interesting and convenient property of the auxiliary network is that for each sub-channel, its all cyclic partition bounds are the same as that of the replicated compound network. To see this, let us fix any sub-channel and consider any cyclic partition bound of the auxiliary network. We show that such a cyclic partition bound has a corresponding cyclic partition in the replicated compound network such that two values are the same. This is because in any cyclic partition bound, the minus value associated with each receiver is only one interference link. Then for any cyclic partition in the auxiliary network, for each receiver, we can find the compound receiver in the replicated network that contains the same interference link. Repeat this process for all receivers, and we arrive at the corresponding state with the matching cyclic partition bound.

As the cyclic partition bound is the same for the auxiliary network and the replicated compound network, finding the tightest compound state of the replicated compound reduces to find the best cyclic partition bound for the auxiliary network, i.e., $\hat{\mathcal{D}}_{\Sigma} = \sum_{m=1}^M \hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$, where $\hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$ is the optimal cyclic partition bound in the m -th sub-channel of the auxiliary network.

To finish the proof that $\sum_{m=1}^M \hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$ is an legal outer bound, we are left to test the applicability of Theorem 1. That is the replicated compound network is TIN optimal, which is easy to see as the original network is TIN optimal, and each sub-channel is invertible, which is assumed in Theorem 3.

We remark that the auxiliary network is only used to find the best cyclic partition bound value of the replicated compound network. The outer bound is proved for the replicated

compound network, instead of the auxiliary network, whose invertibility property is not verified.

Inner Bound: We now proceed to the inner bound and show that $\hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$ is achievable over the m -th sub-channel of the original network.

First, by Theorem 1, we note that $\hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$ is achievable by TIN over the m -th sub-channel of the auxiliary network (note that we do not need invertibility of the auxiliary network for this, only the outer bound direction needs invertibility).

Second, for any $i \in [K]$, each replicated transmitter i and its desired receiver pair in the auxiliary network has the same channel structure. Then due to symmetry, $\hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$ must be achieved by a tuple $(\hat{d}_{1;1}^{[m]}, \dots, \hat{d}_{1;\mu_1}^{[m]}, \hat{d}_{2;1}^{[m]}, \dots, \hat{d}_{K;\mu_K}^{[m]})$ where $\hat{d}_{i;p}^{[m]}$ are the same for all $p \in \{1, \dots, \mu_i\}$, that is each replicated transmitter i gets the same GDoF, i.e., $\hat{d}_{i;p}^{[m]} = \hat{d}_i^{[m]}$. Further, as TIN is used, we can assume each replicated transmitter i use the same power $P^{\hat{\gamma}_i}$ for some $\hat{\gamma}_i \leq 0$.

Finally, we prove that the K -GDoF tuple $(\hat{d}_1^{[m]}, \dots, \hat{d}_K^{[m]})$ is achievable over the m -th sub-channel of the original network such that collectively $\mu_1 \hat{d}_1 + \dots + \mu_K \hat{d}_K \geq \sum_{m=1}^M \sum_{i=1}^K \mu_i \hat{d}_i^{[m]} = \sum_{m=1}^M \hat{\mathcal{D}}_{\Sigma}^{\Pi^{[m]*}}$, which completes the proof. This claim follows by using the same power of the auxiliary network in the original network, i.e., set $\gamma_i = \hat{\gamma}_i$ and Transmitter i uses power P^{γ_i} . As the interfering signals in the original network have the same channel strength as the auxiliary network (only the number of interfering signals are fewer), the achievability of the GDoF tuple is guaranteed.